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Equilibrium Price and Dynamic Virtual Resource Allocation for Wireless Network Virtualization

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Abstract Economic and technical features are equally important to radio resource allocation in wireless network virtualization (WNV). Regarding virtual resource (VR) as commodity, this paper proposes an effective VR allocation scheme for WNV from the perspective of the market-equilibrium theory. First, physical meaning clear utility functions are defined to characterize the network benefits of user equipments (UEs), infrastructure providers (InPs) and virtual network operators (VNOs) in WNV. Then, the VR allocation problem between one InP and multiple VNOs is formulated as a multi-objective optimization problem. To reduce the algorithm complexity, the multiple-objective problem is first decoupled into two single-objective sub-problems. The supplier-layer sub-problem aims to maximize the benefit of the unique InP, while the customer-layer sub-problem aims to maximize the benefits of the multiple VNOs. Both of the separated sub-problems are solved by using standard convex optimization method, and are combined by searching for the equilibrium-price (EP) of the VR market. As a result, the Pareto optimal solution of the original multi-objective problem is found, at which no one (the InP or anyone of the VNOs) can increase its benefit by deviating the EP without hurting others' benefits. The effectiveness of the proposed VR allocation scheme is testified through extensive experiments.

Keywords Wireless network virtualization · Visual resource allocation · Market supply-and-demand theory · Equilibrium price · Pareto optimality

1 Introduction

Due to the popularity of smart UEs, the cellular spectrum is getting more and more congested. Internet service providers (ISPs) require not only new information and communication technologies (ICTs) to improve their spectrum and energy efficiency, but also innovation business model in industry landscape to benefit from the emerging mobile markets. Recently, following ICT resource (e.g., CPU, memory and storage) virtualization in cloud computing [1–3], WNV [4] provides an alternate to achieve both of the above benefits for ISPs. In the context of WNV, the role of a traditional ISP is decoupled into two parts, i.e., the InP part and the VNO part [5]. An InP owns and operates the whole physical substrate and the licensed spectrum resource of a network. These resources are abstracted and isolated into multiple VR slices by the InP. In contrast, a VNO owns none of the resources but can lease VR slices provided by an InP. Thus, a slice of VR holding certain corresponding functionalities is taken as the basic unit for VR allocation [6].

Although network virtualization has occurred in wired networks for decades, the research on WNV is still in its infant stage [7]. Comparing with the wired counterpart, VR

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allocation schemes in WNV face the following challenges [7]. First, VR abstraction and isolation in WNV rely more on physical wireless networks (e.g., 3GPP-LTE, Wi-Fi or WiMAX network) and radio access technologies (e.g., code division multiple access (CDMA) or orthogonal frequency division multiple access (OFDMA) [8]). Second, VR allocation for WNV should be dynamic and flexible, and, hence can respond to the stochastic fluctuation of wireless channel quality.

To the best of our knowledge, there have been only a few works reported on the VR allocation for WNV. In [9, 10], a non-cooperative game model and a bankruptcy game model are respectively proposed to allocate the limited VR slices of a unique InP among multiple competitive VNOs. For the purpose of traffic congestion control, both of the works [9, 10] consider the network scenario where the available VRs at the InP are less than the demand of the VNOs. However, as spectrum resource is denoted by bandwidth (i.e., data rates) directly in [9, 10], the proposed schemes cannot adjust to the time-variant wireless channel quality adaptively. This drawback limits their application in practical WNV. In [11], a combined VR scheduler (including a VR slice scheduler and a user flow scheduler), called network virtualization substrate (NVS), is proposed for WiMAX-based WNV. Although NVS provides effective VR allocation, some important issues require to be further addressed. For example, physical meaning clear utility functions (for UEs), pricing strategies (for InPs), and network benefit functions (for VNOs) are not defined in [11]. In addition, the time-variant channel state information (CSI) as well as the dynamic demand-and-supply variation in the VR market is not well addressed in [11]. Different from [11], the authors in [12, 13] perform the VR allocation on a per subcarrier basis, and the objective is to meet the minimum rate requirements of all UEs while occupying as few resources (which refer to subcarrier in [12] and subcarrier and transmit power in [13]) as possible. Unfortunately, the authors in [12, 13] do not address the economic features of their VR allocation schemes. In [14], the authors consider both the economic and the technical features of VR allocation in WNV. The VR allocation between InPs and VNOs is formulated as a multi-objective optimization problem. However, they do not provide any effective algorithms to solve the problem.

In this paper, we propose an effective VR allocation scheme for OFDMA-based WNV, which considers the time-variant wireless channel quality (from the technical perspective) and the network benefits for UEs, InPs and VNOs (from the economic perspective) jointly. Comparing with existing works, the main contributions of this paper are summarized as follows:

1) Physical meaning clear utility functions are defined for UEs, InPs, and VNOs, respectively. The economic benefits for these WNV components after experiencing a certain level of network service can thus be precisely quantified. The proposed utility functions appeal to the law of diminishing marginal utility [15] and are applicable to the sequel economic and mathematical analysis.
2) In practical applications, an InP or a VNO can serve multiple local UEs. For example, in the UK, O_2 leases the network to GiffGaff while O_2 also has its own subscribers. Considering the InP as well as each of the VNOs is willing to maximize its network benefit through the VR allocation, we formulate the VR allocation problem as a multi-objective optimization problem [16].
3) To solve the multi-objective optimization problem, we resort to the market equilibrium theory [17], and develop an iterative heuristic algorithm to search the EP of the VR market. The algorithm is with low computational complexity and can converge to the EP within 15 time iteratives. At the EP, Pareto optimal VR allocation is achieved, as no one (the InP or one of the VNOs) can increase its benefit by deviating the EP without hurting others' benefits.

For convenience, Table 1 summarizes the main abbreviations and their description used in the following analysis.

The rest of this paper is organized as follows. In Sec. II, we describe the considered OFDMA-based WNV model. Then, in Sec. III, the VR allocation problem is formulated. In Sec. IV, we first introduce the basic of the market-equilibrium theory. Then, low complexity algorithms are developed to solve the VR allocation problem. The simulation results are provided in Sec. IV. Finally, Sec. V concludes the paper.

2 System model

The considered WNV model is shown in Fig. 1. The unique InP abstracts and isolates the infrastructure components and the licensed spectrum resources into multiple VR slices. These VR slices are to be leased to N VNOs (without any network resources). Hence, the VNOs can program on the allocated VR slices and provide wireless services to their own subscribers. We assume that the InP supports M UEs, and the nth ($n = 1, 2 \ldots N$) VNO supports K_n UEs.

We consider that OFDMA technology [8] is used at the network MAC layer. The total W Hz uplink channel bandwidth is divided into S orthogonal subcarriers. Each subcarrier is thus with the bandwidth size $w_0 = W/S$ Hz. A sample MAC frame structure is shown in Fig. 2. The frame is to be transmitted by the UEs in the uplink periodically (typically every 5 ms), and the serving BS can inform the UEs of which subcarriers that they can transmit over in the uplink map of each frame [18]. As this paper focuses on the spectrum resource allocation in WNV, we also assume that if a certain amount of subcarriers is allocated to an InP or a VNO, the

Table 1 Simulation parameters

Abbreviation	Description
WNV	Wireless network virtualization
VR	Virtual resource
UE	User equipment
InP	Infrastructure provider
VNO	Virtual network operator
EP	Equilibrium-price
ISP	Internet service providers

related infrastructure resource is also available to it in the form of a slice of VRs.

In the context of WNV, the period of VR allocation usually spans tens or hundreds frames. Hence, we consider the large-scale fading in wireless channels only. Let P_t denote the transmit power at a transmitter, P_r denote the receiving power at the intended receiver, the channel gain (in large-scale) from the transmitter to the receiver can be expressed as [19]

$$\frac{P_t}{P_r}\mathrm{dB} = 10\log(L_0) + 10\kappa\log\left(\frac{d}{d_0}\right) - \psi \tag{1}$$

where $L_0 < 1$ denotes the free-space gain at reference distance d_0, d denotes the distance between the transmitter and the receiver, κ denotes the path-loss exponent and ψ is a Gauss-distributed random variable with zero mean and variance σ_ψ^2. ψ represents the shadowing of wireless channel. ψ characterizes the spatial variation in signal attenuation for the same distance from transmitter, which usually follows a log-normal distribution.

Denote the channel gain from the mth UE of the InP to the BS by g_m. Denote the channel gain from the kth ($k = 1, 2 \ldots K_n$) UE of the nth VNO to the BS by $g_{n,k}$. We assume that each subcarrier experiences frequency-flat fading, i.e., the channel gains remain constant during one VR allocation period. Let p_m denote the transmit power at the mth UE of the InP. Let $p_{n,k}$ denote the transmit power at the kth UE of the nth VNO. The UEs will spread the power on the assigned subcarriers for uplink transmissions. Let c_m ($1 \le c_m < S$) denote the number of subcarriers assigned to the mth UE of the InP. Let $c_{n,k}$ ($1 \le c_{n,k} < S$) denote the number of subcarriers assigned to the kth UE of the nth VNO. Their achievable data rates can be approximated by

$$R_m = c_m w_0 \log_2\left(1 + \frac{p_m \cdot g_m}{c_m w_0 n_0}\right),\ m = 1, 2, \ldots, M \tag{2}$$

and

$$R_{n,k} = c_{n,k} w_0 \log_2\left(1 + \frac{p_{n,k} \cdot g_{n,k}}{c_{n,k} w_0 n_0}\right),\ n = 1, 2, \ldots, N,\ k = 1, 2, \ldots, K_n \tag{3}$$

respectively, where the VR allocation period is normalized to one, and the spectrum density of the noise power at the BS is assumed to be $n_0/2$.

3 Problem formulation

The objective of VR allocation in WNV usually has the following two folds. The first is to improve the spectrum and energy utilization efficiencies from a technical point of view, and the second is to benefit InPs and VNOs from an economic point of view. In commercial networks, the economic benefit received by a UE after experiencing a certain level of network service can be quantified by using *utility functions*. In

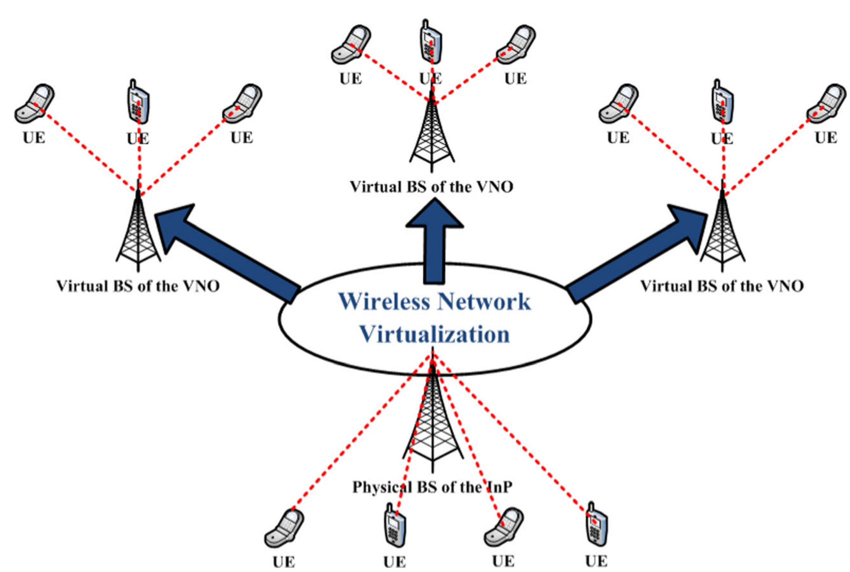

Fig. 1 The considered WNV model

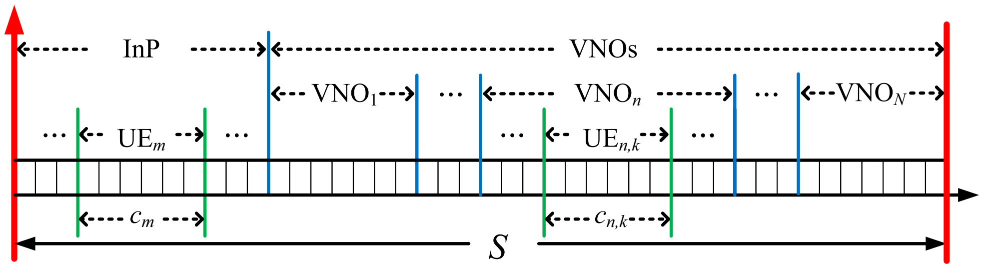

Fig. 2 The VR allocation pattern

literatures, several types of *utility functions* have been proposed, e.g., the *logarithmic* type *utility functions* for data-rate sensitive applications [19, 20] and the *exponential* type *utility functions* for delay sensitive applications [21, 22]. Based on Eqs. (2) and (3), we define the utility functions for the UEs as

$$U_m(R_m) = \ln R_m = \ln\left(c_m w_0 \log_2\left(1 + b_m \frac{p_m \cdot g_m}{c_m w_0 n_0}\right)\right), \ m = 1, 2, \ldots, M \tag{4}$$

and

$$U_{n,k}(R_{n,k}) = \ln R_{n,k} = \ln\left(c_{n,k} w_0 \log_2\left(1 + b_{n,k} \frac{p_{n,k} \cdot g_{n,k}}{c_{n,k} w_0 n_0}\right)\right), \ n = 1, 2, \ldots, N, \ k = 1, 2, \ldots, K_n \tag{5}$$

respectively.

Given $c_m \geq 0$ (or $c_{n,k} \geq 0$), we can prove that $U_m(R_m)$ (or $U_{n,k}(R_{n,k})$) increases monotonically with c_m (or $c_{n,k}$), and, $\partial U_m/\partial c_m$ (or $\partial U_{n,k}/\partial c_{n,k}$) decreases monotonically with c_m (or $c_{n,k}$). This implies that the proposed utility functions meet the law of *diminishing marginal utility* [15]. Hence, the utility of a UE is monotonically increasing with the achievable rate, but each subsequent unit of rate is valued less than the previous one.

As the InP is the unique VR provider, its benefit function consists of the following two parts. The first part is the total utilities gained by serving the M UEs, and the second part is the revenue received from selling the VR slices to the VNOs. Let α denote the price of unit VR (equivalent to an OFDMA subcarrier in this paper) charged by the InP. The benefit function of the InP in the VR market is given by

$$\pi_0 = \sum_{m=1}^{M} U_m + \alpha \sum_{n=1}^{N} C_n \tag{6}$$

where $C_n = \sum_{k=1}^{K_n} c_{n,k}$ is the number of subcarriers sold by the InP to the nth VNO.

Similarly, for the nth VNO, its benefit function consists of the total utilities gained by serving the K_n UEs minus the cost for purchasing a certain amount of VR slice from the InP. Hence, the benefit function for the nth VNO in the VR market is given by

$$\pi_n = \sum_{k=1}^{K_n} U_{n,k} - \alpha \cdot C_n, \ n = 1, 2 \ldots N \tag{7}$$

Based on the above analysis, we can formulate the VR allocation problem as

$$\max(\pi_0, \pi_1, \ldots, \pi_N) \tag{8}$$

$$\text{s.t. } c_m \in \{1, 2, \ldots, S-1\}, \ m = 1, \ldots, M \tag{8.1}$$

$$c_{n,k} \in \{1, 2, \ldots, S-1\}, \ n = 1, \ldots, N, \ k = 1, \ldots, K_n \tag{8.2}$$

$$\sum_{n=1}^{N} C_n + \sum_{m=1}^{M} c_m \leq S \tag{8.3}$$

Constraints (8.1) and (8.2) indicate that the feasible space for problem (8) is discrete. Constraint (8.3) indicates that the demanded VR resource is not more than the available VR resource.

Remark 1: By observing problem (8), we know that:

(1) Problem (8) is an integer-programming problem [23], as parameters c_m and $c_{n,k}$ to be optimized are in terms of integers. There exists no polynomial time-complexity algorithm to solve this type of integer-programming problem [23].
(2) Problem (8) is a multi-objective optimization problem. The solution to such type of problems is not unique [24].

To address the first difficulty, we can relax constraints (8.1) and (8.2) into continuous ones as in [24] by allowing c_m and $c_{n,k}$ to be real numbers within $[1, S-1]$. To address the second difficulty, we will apply the following *Pareto optimality* [16] to filter the solutions of problem (8).

Definition 1: Within the solution space, a VR allocation vector is said to be *Pareto optimum* if there exists no other such vectors by which one player (the unique InP or one of the VNOs) can increase its benefit without decreasing the benefits of the other players.

4 Dynamic VR allocation in WNV

The relaxed version of problem (8) with continuous constraints is given as

$$\max(\pi_0, \pi_1, \ldots, \pi_N) \tag{9}$$

$$\text{s.t. } 0 < c_m < S,\ m = 1, \ldots, M \tag{9.1}$$

$$0 < c_{n,k} < S,\ n = 1, \ldots, N,\ k = 1, \ldots, K_n \tag{9.2}$$

$$\sum_{n=1}^{N} C_n + \sum_{m=1}^{M} c_m \leq S \tag{9.3}$$

However, problem (9) is still a multi-objective problem. To solve this difficulty, we resort to the market equilibrium theory [25]. The solution detail is given as follows:

1) Transform problem (9) into two independent single-objective problems. The supplier-layer problem is with the objective to maximize the benefit of the unique InP, while the customer-layer problem is with the objective to maximize the benefits of the multiple VNOs.
2) By formulating the single-objective problems as strict convex problems, they can be solved by using standard convex optimization method.
3) The two interrelated single-objective problems are combined via searching for the EP of the VR market. As a result, the Pareto optimal solution of the original problem (9) can be found.

4.1 Basic of the market equilibrium theory

In commercial networks, e.g., WNV, VR is usually regarded as commodity. Pricing VR is a common way to coordinate VR allocation among InPs and VNOs. In microeconomics, the supply-and-demand equilibrium model is an effective tool to investigate the price of a particular commodity and the quantity of that commodity which is traded in a market. It is normal to take the quantity demanded by a consumer and the quantity supplied by a supplier as a function of the price of the commodity. The standard graphical representation of the supply and demand equilibrium model is illustrated in Fig. 3, where the vertical axis represents the price and the horizontal axis represents the quantity.

In Fig. 3, the supply curve represents the amount of a commodity that a supplier is willing and able to sell at various prices, and the demand curve represents the amount of a commodity that a consumer is willing to purchase at various prices. A negotiation on the price and the quantity of that commodity is required between the supplier and the consumer to ensure that both are satisfied with the solution. The intersection of the demand and the supply curves is referred to as an EP of the market. An EP is formally defined as the price-quantity pair where the quantity demanded is equal to the quantity supplied. According to the *first welfare* theorem [25], if a perfectly competitive market is in full equilibrium, the EP leads to a Pareto optimal outcome for both the consumer and the supplier.

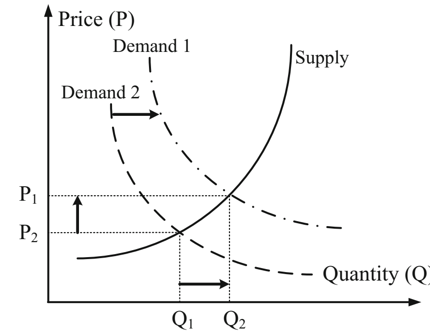

Fig. 3 The supply and demand equilibrium model

4.2 Supplier-level problem

Given the market price α, the InP can calculate the optimal amount of VR that is willing to supply by solving the following problem

$$\max \pi_0(\alpha),\ \pi_0(\alpha) = \sum_{m=1}^{M} U_m + \alpha\left(S - \sum_{m=1}^{M} c_m\right) \tag{10}$$

$$\text{s.t. } 0 < c_m,\ m = 1, \ldots, M \tag{10.1}$$

$$c_m < S,\ m = 1, \ldots, M \tag{10.2}$$

$$\sum_{m=1}^{M} c_m < S \tag{10.3}$$

Remark 2: By examining problem (10), we know that

1) The problem is a single-objective optimization problem.
2) The constraint set is affine as it is composed of linear constraints.
3) The first term of the objective function, i.e., $\sum_{m=1}^{M} U_m$, is concave for $1 \leq c_m \leq S-1$, since it is a sum of concave functions.

Based on the above observations, the concavity of the objective function $\pi_0(\alpha)$, and, hence, the existence of a unique global solution to problem (10), depends only on the region of price α. To simplify the notation, we define

$$G_m = \frac{p_m g_m}{w_0 n_0},\ m = 1, \ldots, M \tag{11}$$

G_m is a constant. The following Proposition gives the largest value of α which ensures the concavity of $\pi_0(\alpha)$.

Proposition 1: To ensure the strict concavity of $\pi_0(\alpha)$, the highest price that the InP can charge is given as

$$\alpha = \min(\chi_1, \dots, \chi_M) \tag{12}$$

where

$$\chi_m = \frac{\left(1+\frac{c_m}{G_m}\right)\ln\left(1+\frac{G_m}{c_m}\right)-1}{c_m\cdot(G_m+c_m)\ln\left(1+\frac{G_m}{c_m}\right)} > 0,\ m = 1, \dots, M \tag{13}$$

Proof: The proof is given in Appendix A.

When Proposition 1 is satisfied, problem (10) is strict convex. The unique optimal solution to problem (10) is given in the following proposition.

Proposition 2: On condition that Proposition 1 is satisfied, the unique solution to problem (10) is given as

$$c_m^* = \arg_{c_m}\left((1-\alpha G_m c_m)\ln\left(1+\frac{G_m}{c_m}\right)=1\right),\ m = 1, \dots, M \tag{14}$$

Proof: The proof is given in Appendix B.

4.3 Customer-level problem

Similar to the supplier-level problem, given the current market price α, the N VNOs can calculate the optimal amount of VR that they could purchase by solving the following problem

$$\max(\pi_1(\alpha), \pi_2(\alpha), \dots, \pi_N(\alpha)) \tag{15}$$

$$\text{s.t. } 0 < c_{n,k},\ n = 1, \dots, N,\ k = 1, \dots, K_n \tag{15.1}$$

$$c_{n,k} < S,\ n = 1, \dots, N,\ k = 1, \dots, K_n \tag{15.2}$$

$$\sum_{n=1}^{N} C_n < S \tag{15.3}$$

However, problem (15) is still a multi-objective problem, and there exists multiple solutions to the problem. To address this difficulty, we give the following Proposition.

Proposition 3: Given the market price α charged by the InP, if the Pareto optimality is explicitly imposed, the solution to the following problem (16) yields the same solution to problem (15).

$$\max \Omega(\alpha),\ \Omega(\alpha) = \sum_{n=1}^{N} \pi_n(\alpha) \tag{16}$$

$$\text{s.t. } 0 < c_{n,k},\ n = 1, \dots, N,\ k = 1, \dots, K_n \tag{16.1}$$

$$c_{n,k} < S,\ n = 1, \dots, N,\ k = 1, \dots, K_n \tag{16.2}$$

$$\sum_{n=1}^{N} C_n < S \tag{16.3}$$

Proof: The proof is given in Appendix C.

Note that Proposition 3 only provides a sufficient condition, and the reverse necessity is not necessarily true. Similar to the supplier-level problem (10), the concavity of the objective function $\Omega(\alpha)$ of problem (16) only depends on the value of α. To simplify the notation, we define

$$G_{n,k} = \frac{w_0 n_0}{p_{n,k} g_{n,k}},\ n = 1, \dots, N,\ k = 1, \dots, K_n \tag{17}$$

$G_{n,k}$ is a constant. The following Proposition gives the largest value of α which ensures the concavity of $\Omega(\alpha)$.

Proposition 4: To ensure the strict concavity of $\Omega(\alpha)$, the highest price that the InP can charge is given as

$$\alpha = \min\{\chi_{n,k} | n = 1, \dots, N,\ k = 1, \dots, K_n\} \tag{18}$$

where

$$\chi_{n,k} = \frac{\left(1+\frac{c_{n,k}}{G_{n,k}}\right)\ln\left(1+\frac{G_{n,k}}{c_{n,k}}\right)-1}{c_{n,k}\cdot\left(G_{n,k}+c_{n,k}\right)\ln\left(1+\frac{G_{n,k}}{c_{n,k}}\right)} > 0,\ n = 1, \dots, N,\ k = 1, \dots, K_n \tag{19}$$

Proof: The proof is similar to that for Proposition 1. Thus, it is omitted here.

When the condition in Proposition 4 is satisfied, problem (16) is strict convex, and the unique optimal solution to problem (16) is given by the following proposition.

Proposition 5: On condition that Proposition 4 is met, the unique solution to problem (16) can be solved as

$$c_{n,k}^* = \arg_{c_{n,k}}\left((1-\alpha G_{n,k} c_{n,k})\ln\left(1+\frac{1}{A_{n,k} c_{n,k}}\right)=1\right),\ n = 1, \dots, N,\ k = 1, \dots, K_n \tag{20}$$

Proof: The proof is similar to that for Proposition 2. Thus, it is omitted here.

4.4 Equilibrium price of the VR market

Given the current VR price α, the supplier-level problem (10) and the customer-level problem (16) can be solved independently. In this section, we develop an iterative method to search the EP of the VR market, which results in Pareto optimal solution to problem (9).

In the supply and demand market model [25], if the demand is higher than the supply, the supplier can increase price to obtain higher revenue. On the contrary, if the supply is higher than the demand, the supplier can decrease price to attract more consumption. According to this rule, we develop an iterative price update function to search the EP as follows

$$\alpha(t+1) = \alpha(t) + \lambda\left(\Phi^{\mathrm{D}}(t) - \Phi^{\mathrm{S}}(t)\right) \tag{21}$$

where t ($t = 0, 1, 2, \ldots$) represents the iterative number, $\Phi^{\mathrm{D}}(t) = \sum_{n=1}^{N}\sum_{k=1}^{K_n} c_{n,k}(t)$ (which represents the demanded quantity) is obtained by solving problem (16), $\Phi^{\mathrm{S}}(t) = S - \sum_{m=1}^{M} c_m^t(t)$ (which represents the supplied quantity) is obtained by solving problem (10), and $\Phi^{\mathrm{D}}(t) - \Phi^{\mathrm{S}}(t)$ represents the excess demand. At each iterative round t, the excess demand is weighted by the speed adjustment parameter λ, and is added to the current price for price updating. The update is repeated until the stopping criterion

$$|\alpha(t+1) - \alpha(t)| < \varepsilon,\ \varepsilon > 0, \tag{22}$$

is met, where ε is an arbitrarily small positive. The detailed algorithm to search the EP is shown in Algorithm 1.

Algorithm 1: The algorithm to search the EP of the VR market

Step 0 (Initialization): Let $t = 0$. The InP sets the initial price $\alpha(t) = \alpha^{\mathrm{ini}}$.

Step 1: Given the current VR price $\alpha(t)$, the following parameters can be determined:

(1) By solving problem (10), the optimal VR allocation vector $c_m^t(t)$, $\forall m = 1, \ldots, M$, for the InP, and, hence $\Phi^{\mathrm{S}}(t)$ can be obtained;

(2) By solving problem (16), the optimal VR allocation vector $c_{n,k}(t)$, $\forall n = 1, \ldots, N$, $\forall k = 1, \ldots, K_n$ for the N VNOs, and, hence $\Phi^{\mathrm{D}}(t)$ can be obtained;

Step 2: The excess demand at iterative round t is calculated as $\Phi^{\mathrm{D}}(t) - \Phi^{\mathrm{S}}(t)$, and the VR price is updated by using Eq. (21).

Step 3: If criterion (22) is satisfied, the algorithm terminates. Otherwise, let $t = t+1$, and go back to *Step 1*.

4.5 Optimal solution

Up to this point, the discrete constraints (8.1) and (8.2) are not considered yet. The common method to deal with this kind of problems is mapping the continuous solution to the largest previous integer [19] by the following operation

$$\overline{c}_m^* = \lfloor c_m^* \rfloor,\ m = 1, \ldots, M \text{ and } \overline{c}_{n,k}^* = \left\lfloor c_{n,k}^* \right\rfloor,\ n = 1, \ldots, N,\ k = 1, \ldots, K_n \tag{23}$$

where $\lfloor x \rfloor$ denotes the largest integer no more than x. After the above operation, the remaining subcarriers at the InP and at the nth VNO can be given as

$$\overline{C}_{\mathrm{InP}} = \sum_{m=1}^{M}\left(c_m^* - \overline{c}_m^*\right) \text{ and } \overline{C}_n = \sum_{k=1}^{K_n}\left(c_{n,k}^* - \overline{c}_{n,k}^*\right),\ n = 1, \ldots, N \tag{24}$$

respectively, where

$$\overline{C}_{\mathrm{InP}} \geq 0 \text{ and } \overline{C}_n \geq 0 \text{ for } \forall n = 1, \ldots, N \tag{25}$$

It is ordered that the remaining VR should be allocated to the VNOs in a Pareto improvement manner. For that purpose, the following two heuristics should be performed in sequence.

Algorithm 2: The VR reallocation algorithm for the VNOs

Step 0 (Initialization): Let $\tilde{C}_n = \overline{C}_n$ and $\tilde{c}_{n,k} = \overline{c}_{n,k}^*$ for $n = 1, \ldots, N$, $k = 1, \ldots, K_n$;

Step 1: **For each nth VNO**

Step 2: **While** $\tilde{C}_n \geq 1$

Step 3: $\tilde{k} = \arg\max_{k=1,\ldots,K_n} \Delta_{n,k}$, $\Delta_{n,k} = U_{n,k}\left(\tilde{c}_{n,k} + 1\right) - U_{n,k}\left(\tilde{c}_{n,k}\right)$;

Step 4: $\tilde{c}_{n,k} = \tilde{c}_{n,k} + 1$;

Step 5: $\tilde{C}_n = \tilde{C}_n - 1$;

Step 6: **End While**

Step 7: **End For**

Step 8: $\tilde{C}_{\mathrm{VNO}} = \sum_{n=1}^{N} \tilde{C}_n$;

Step 9: **While** $\tilde{C}_{\mathrm{VNO}} \geq 1$

Step 10: $(\tilde{n}, \tilde{k}) = \arg\max_{n=1,\ldots,N, k=1,\ldots,K_n} \Delta_{n,k}$, $\Delta_{n,k} = U_{n,k}\left(\tilde{c}_{n,k} + 1\right) - U_{n,k}\left(\tilde{c}_{n,k}\right)$

Step 11: $\tilde{c}_{n,k} = \tilde{c}_{n,k} + 1$;

Step 12: $\tilde{C}_{\mathrm{VNO}} = \tilde{C}_{\mathrm{VNO}} - 1$;

Step 13: **End While**

Step 14: **Return** $\tilde{c}_{n,k}$, $k = 1, \ldots, K_n$, $n = 1, \ldots, N$, and $\tilde{C}_{\mathrm{VNO}}$.

Remark 3: The *Steps 1* to *7* in algorithm 2 can allocate the remaining $\overline{C}_n$ subcarriers to the UEs (served by the nth VNO) in a Pareto improvement manner. The intuition is explained as follows. In *Step 3*, each kth UE served by the nth VNO announces its marginal utility Δ_k. Then, in *Step 4*, a remaining subcarrier can be allocated to the $\tilde{k}$ th UE with the maximal marginal utility. As the UE with the lowest utility usually has the highest marginal utility, the algorithm can improve the utilities of the served UEs in the Pareto improvement manner. In addition, the algorithm can also bring maximal benefit for the VNO, since the remaining $\overline{C}_n$ subcarriers of the nth VNO are allocated to the UEs one by one to maximize the utility argument per subcarrier.

Table 2 Simulated network cases

	The number of the UEs served by the InP, M	The number of the VNOs, N	The number of the UEs served by the nth VNO, K_n
Case 1	3	2	3
Case 2	3	3	4
Case 3	5	2	3
Case 4	5	3	4

After performing the VR reallocation at each VNO, there are still $\tilde{C}_{\mathrm{VNO}} = \sum_{n=1}^{N} \tilde{C}_n$ $(0 < \tilde{C}_{\mathrm{VNO}} < N)$ subcarriers available. *Step 9* to *13* of the algorithm 2 is used to further allocate these $\tilde{C}_{\mathrm{VNO}}$ subcarriers to the UEs in the Pareto improvement manner. Although there are still $\overline{C}_{\mathrm{VNO}}$ subcarriers available after performing *Step 9* to *13*, obviously, $0 < \overline{C}_{\mathrm{VNO}} < 1$. It means that the proposed VR reallocation algorithm will not affect the EP so much.

Algorithm 3: The VR reallocation algorithm for the InP

Step 0 (Initialization): $\tilde{C}_{\mathrm{InP}} = \overline{C}_{\mathrm{InP}} + \tilde{C}_{\mathrm{InP}}$, $\tilde{c}_m = \overline{c}_m^*$, $n = 1, \ldots, N$, $k = 1, \ldots, K_n$;

Step 1: **While** $\tilde{C}_{\mathrm{InP}} \geq 1$

Step 2: $\tilde{m} = \arg \max_{m=1,\ldots,M} \Delta_m$, $\Delta_m = U_m(\tilde{c}_m + 1) - U_m(\tilde{c}_m)$;

Step 3: $\tilde{c}_m = \tilde{c}_m + 1$;

Step 4: $\tilde{C}_{\mathrm{InP}} = \tilde{C}_{\mathrm{InP}} - 1$;

Step 5: **End While**

Step 6: **Return** $\tilde{c}_m$, $k = 1, \ldots, K_n$, $n = 1, \ldots, N$.

Remark 4: In *Step 0* of Algorithm 3, the InP first recycles the $\tilde{C}_{\mathrm{InP}}$ subcarriers which cannot be utilized by the VNOs. Then the total $\tilde{C}_{\mathrm{InP}} = \overline{C}_{\mathrm{InP}} + \tilde{C}_{\mathrm{VNO}}$ subcarriers need to be allocated to the M UEs. For that purpose, the InP allocate these subcarriers to the UEs with the maximal marginal utility in *Step 1* to *5*. Therefore, the benefit of the InP and the utilities of the served UEs can be increased in the Pareto improvement manner. It is noted that $\tilde{C}_{\mathrm{InP}}$ is an integer and $0 \leq \tilde{C}_{\mathrm{InP}} - \overline{C}_{\mathrm{InP}} \leq 1$

Table 3 Simulation parameters

Parameter	Value
Noise PSD $n_0/2$	5×10^{-15}
Transmission power of a UE	0.2 W
Subcarrier bandwidth	50 KHz
System bandwidth	1.25 MHz
Reference distance d_0	0.1 km
Path-loss exponent κ	3.71
Variance of ψ	1 dB
Free-space gain at distance d_0 $10\log_{10}(L_0)$	—31.54 dB

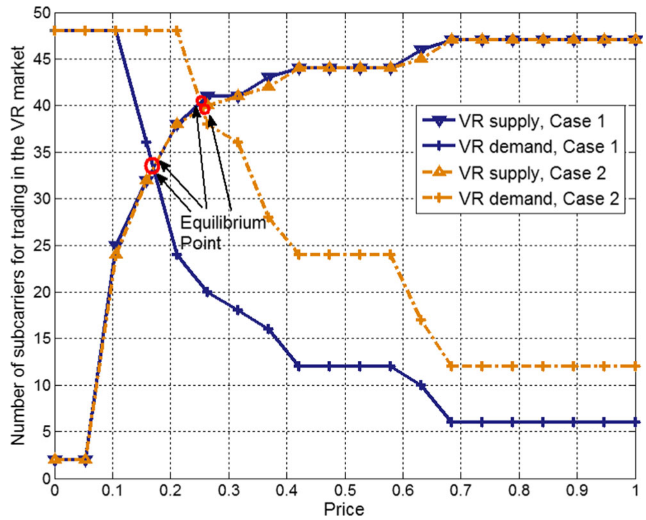

Fig. 4 The supply-and-demand curves in Cases 1 and 2

as $0 < \overline{C}_{\mathrm{VNO}} < 1$. So the proposed VR reallocation algorithm 3 will not affect the EP so much.

4.6 Complexity analysis

In the context of WNV, all the proposed algorithms 1, 2 and 3 can be performed at the network virtualization controller (NVC) in a centralized manner. At each iterative round t, all the required information includes the current VR price $\alpha(t)$ and the global CSI of the UEs. Via dedicated feedback channel, e.g., the cognitive pilot channel (CCC) proposed by the E2R2/E3 consortium in [26], these information can be conveyed between the NVC and the distributed UEs reliably.

In Algorithm 1, we note that parameter λ in Eq. (21) has a significant impact on the convergence speed. If λ is too large, excess-demand would result in the fluctuation on the price adjustment. Consequently, the EP may not be reached by the algorithm. The following simulation will show that Algorithm

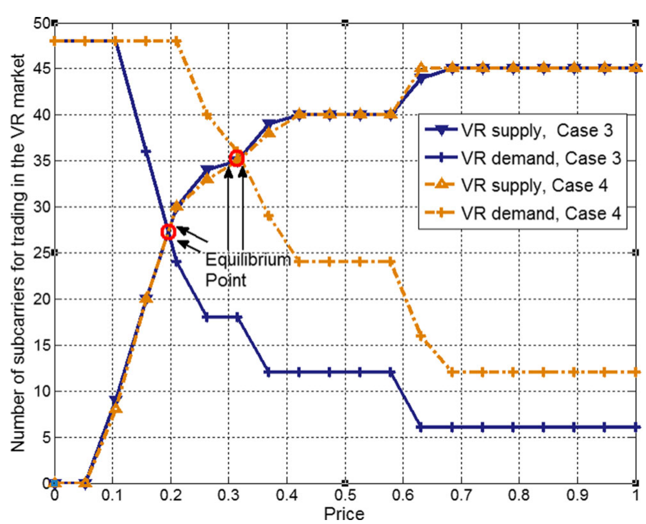

Fig. 5 The supply-and-demand curves in Cases 3 and 4

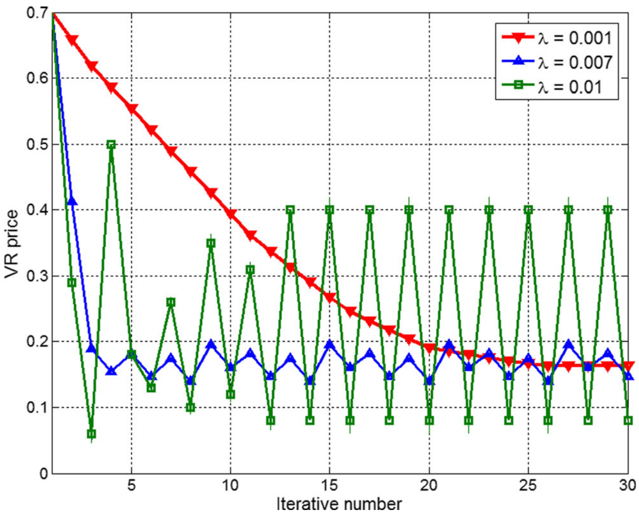

Fig. 6 The convergence of Algorithm 1 in Case 1

1 is stable and can converge to the EP after several (less than 15) times iterative computations for selected λ.

As Algorithm 2 and Algorithm 3 both execute non-iterative process, the computational complexities of the algorithms are $O(NS)$ and $O(S)$, respectively, where N is the total number of the VNOs and S is the total number of the available subcarriers.

5 Numerical results

5.1 Equilibrium price of the VR market

First, to testify the effectiveness of the market equilibrium theory, we show the existence of the EP in the VR market. For that purpose, we locate the unique physical BS of the InP at coordinate (0, 0), and the following four network cases, as shown in Table 2 are considered.

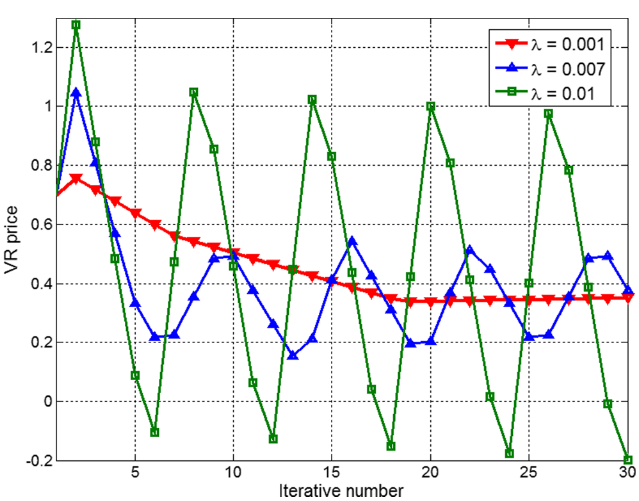

Fig. 7 The convergence of Algorithm 1 in Case 4

Table 4 Simulated network cases

	Number of subcarriers allocated to the InP	Number of subcarriers allocated to the 1st VNO	Number of subcarriers allocated to the 2nd VNO	Number of subcarriers allocated to the 3rd VNO
Case 1	16	17	17	NA
Case 2	10	13	13	14
Case 3	24	13	13	NA
Case 4	15	11	12	12

All the UEs are uniformly distributed in a circle area with radius $r = 1$ km, and the circle center is also at coordinate (0,0). The unit is kilometer. The other parameters used in the simulations are shown in Table 3.

In Figs. 5 and 6, we show the VR supply curve and the VR demand curve in different network cases, respectively. We can observe that the demand of the VNOs decreases and the supply of the InP increases, along with the increasing VR price. At each intersection-point of a supply-curve and a demand-curve, the VR supply just equals to the VR demand. The intersection-points are the EPs for the different network cases.

From Fig. 4, we can also observe that the VR supplies remain almost the same in Cases 1 and 2 at different prices. It is because the numbers of the UEs served by the InP are the same in Cases 1 and 2. As N increases from 2 to 3, and K_n increases from 3 to 4, the VR demand of the VNOs increases significantly when the VR price is higher than 0.1. This is because, as N and K_n increase, the VNOs tend to lease more VR from the InP to improve the utilities of their UEs. Hence, they can earn more network profits. Similar results of Cases 3 and 4 can be observed in Fig. 5.

Remark 5: From the analysis above, we can conclude that the proposed VR allocation scheme can effectively coordinate the VR supply-and-demand between the InP and the VNOs. The InP should charge the VNOs the EP, so as to obtain the Pareto optimal VR allocation. Of course, any other price strategies can also be proposed by applying different resource allocation methods, e.g., the non-cooperative game theory [7] and the cooperative game theory [10, 18]. In our proposed scheme, the EP α is not only an algorithm parameter to coordinate the VR allocation among the InP and the VNOs, but also has practical significance as it is introduced according to the framework of economic analysis. The obtained EP α is an intuitive factor for an InP to develop the practical pricing strategy.

5.2 Convergence of the proposed algorithm

In this section, we show the convergence property of the proposed iterative price adjustment algorithm, i.e., Algorithm 1. We examine Case 1 and Case 4 specified in Section 5. A. In both the cases, the iterative stopping criterion in Eq. (22) is

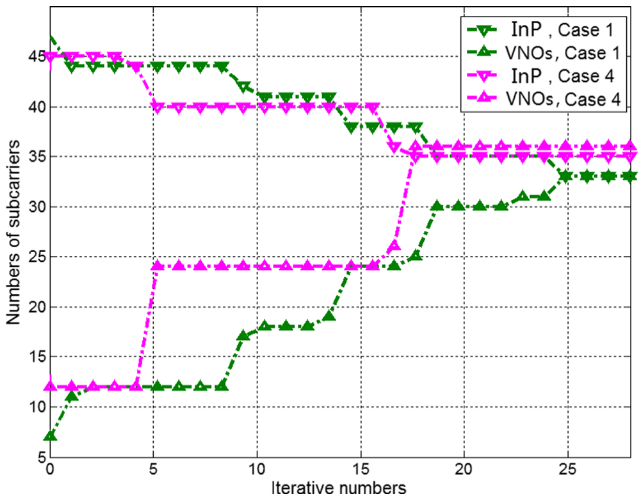

Fig. 8 The coincidence of supply curves and demand curves

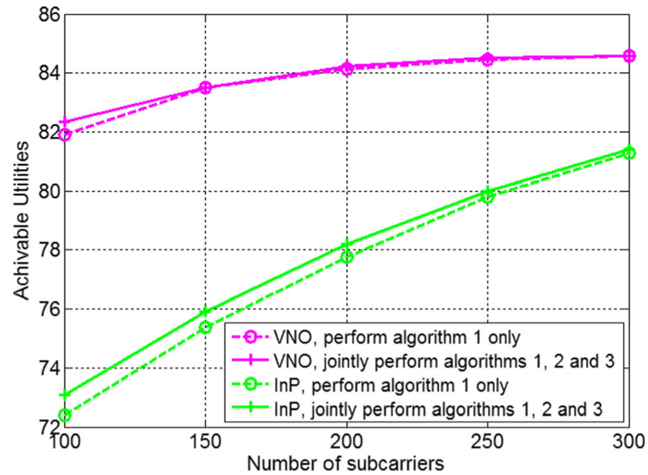

Fig. 10 The achievable utilities of the InP and the VNO by performing different algorithms

$\varepsilon = 10^{-3}$, and the initial price charged by the InP is $\alpha(0) = 0.7$. For different speed adjustment parameter λ, the convergences of Algorithm 1 in Cases 1 and 4 are shown in Figs. 6 and 7, respectively.

From Fig. 4, we know that the EP in Case 1 is about 0.17. When $\lambda = 1 \times 10^{-3}$, the EP can be approached by less than 30 times of iterative computation as shown in Fig. 6. However, when the speed adjustment parameter λ is with larger values, e.g., $\lambda = 7 \times 10^{-3}$, the system is unstable. Algorithm 1 shows a shock behavior in the network case. The similar convergence and divergence behaviors of Algorithm 1 in Case 4 can also be observed in Fig. 7. In this paper, we use the "iterative method" to find the equilibrium price of the VR market. The iterative method uses successive approximations to obtain more accurate solutions to the liner system, composed of Eq. (21) and (22) at each step. When attempting to solve the liner system by finding successive approximations of λ starting from an initial guess, the iterative method can only arrive at a satisfied approximation based on a measurement of the error in the result, and form a "correction equation" for which this process is repeated. Although the method is simple to implement, convergence is only guaranteed for a limited class of matrices as shown in Figs. 6 and 7.

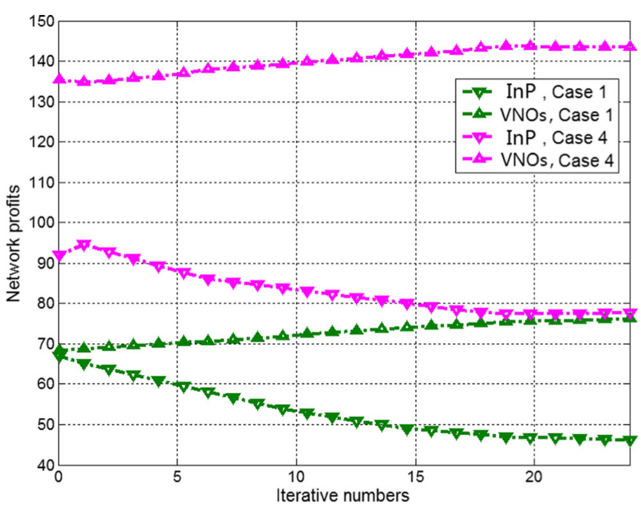

Fig. 9 The benefits convergence of the InP and the VNOs

5.3 Pareto optimality of the VR allocation

First, in Table 4, we show the VR (i.e., subcarrier) allocation results for the InP and the VNOs in different network cases after performing Algorithms 1, 2 and 3. Note that all the results are obtained when the EPs are achieved in different network cases.

Then, we testify the Pareto optimality of the VR allocation results. For that purpose, we set $\lambda = 1 \times 10^{-3}$, $\varepsilon = 10^{-3}$ and $\alpha(0) = 0.7$ for network cases 1 and 4. In Fig. 8, we show the convergences of the supplies (of the InP) and the demands (of the VNOs), and, in Fig. 9, we show the resultant benefit convergences of the InP and the VNOs.

From Fig. 8, we observe that, with initial price $\alpha(0) = 0.7$, there exist significant differences between the supplies and demands in the VR market. As the VR price converges to the EP, the supply curves and the demand curves gradually coincide with each other. Jointly considering the benefit convergence of the InP and the VNOs as shown in Fig. 9, we can conclude that at an EP, the VR allocation is Pareto optimality, as no one (the InP as well as anyone of the VNOs) can increase its benefit by deviating the EP without hurting others' benefits.

Finally, we show the performance of the proposed algorithms 2 and 3, which can further increase the achievable utilities for the InPand the VNOs. In the following simulation, we assume that there is one VNO requesting VR from the InP. The number of UEs associated with the InP is $M = 5$, and the number of UEs associated with the VNO is $K_1 = 6$. We increase the number of subcarriers available to the InP and the VNO from $S = 100$ to $S = 300$ at a step of 50. The achievable utilities for the InP and the VNO by performing different algorithms are shown in Fig. 10.

From Fig. 10, we can observe that by performing the proposed algorithms 2 and 3, the achievable utilities of the InP and the VNO can be further increased. However, there is a diminishing effect of the reallocation algorithms 2 and 3 if the number of the subcarriers is large.

6 Conclusions

In this paper, a market equilibrium theory based VR allocation scheme is proposed for OFDMA-based WNVs. Considering the InP and the VNOs are willing to maximize its own benefit, the VR allocation problem is formulated as a multi-objective optimization problem. Decouple this multi-objective problem into two independent single-objective problems. Both of them can be solved by using standard convex optimization methods. By searching the EP of the VR market, the original multi-objective problem is finally solved in the Pareto optimal sense. The effectiveness of the proposed VR allocation scheme has also been testified through extensive experiments.

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Appendix A

Proof of proposition 1

Taking the first- and second- order derivatives of $\pi_0(\alpha)$ with respect to c_m, we have

$$\frac{\partial \pi_0(\alpha)}{\partial c_m} = \frac{\partial U_m}{\partial c_m} - \alpha, \ m = 1, \ldots, M \tag{26}$$

and

$$\frac{\partial^2 \pi_0(\alpha)}{\partial c_m^2} = \frac{\partial^2 U_m}{\partial c_m^2} < 0, \ m = 1, \ldots, M \tag{27}$$

In order to guarantee the concavity of $\pi_0(\alpha)$, it is ordered to derive the condition under which $\partial \pi_0(\alpha)/\partial c_m > 0$. By substituting Eq. (4) into Eq. (26), the condition is derived as shown in Eqs. (12) and (13).

Appendix B

Proof of proposition 2

To solve problem (10), we exploit the Karush-Kuhn-Tucker (KKT) optimality condition [21]. The Lagrangian function of problem (10) is written as

$$\begin{aligned} L_0(\alpha) &= \sum_{m=1}^{M} U_m + \alpha \left(S - \sum_{m=1}^{M} c_m \right) \\ &+ \mu \left(\sum_{m=1}^{M} c_m - S \right) - \sum_{m=1}^{M} \nu_m c_m + \sum_{m=1}^{M} \rho_m (c_m - S) \end{aligned} \tag{28}$$

where μ, ν_m and ρ_m, $\forall\, m = 1, \ldots, M$, are the non-negative Lagrangian multipliers. The KKT optimality conditions for this problem are:

$$\begin{aligned} \frac{\partial L_0(\alpha)}{\partial c_m} &= \frac{\ln\left(1 + \frac{G_m}{c_m}\right) - 1}{c_m G_m \ln\left(1 + \frac{G_m}{c_m}\right)} - \alpha + \mu - \nu_m + \rho_m \\ &= 0, \ m = 1, \ldots, M \end{aligned} \tag{29}$$

with the additional complementary slackness conditions as:

$$\mu \left(\sum_{m=1}^{M} c_m - S \right) = 0 \tag{30}$$

$$\rho_m (c_m - S), \ m = 1, \ldots, M \tag{31}$$

$$\nu_m c_m = 0, \ m = 1, \ldots, M \tag{32}$$

Since $0 < c_m < S$ and $\sum_{m=1}^{M} c_m < S$ for $\forall\, m = 1, \ldots, M$, constraints (10.1), (10.2) and (10.3) are all satisfied without equality. So we get $\nu_m = 0$, $\rho_m = 0$ and $\mu = 0$. Then, by solving Eq. (29), we can obtain the optimal solution to problem (10) as shown in Eq. (14).

Appendix C

Proof of proposition 3

First, we assume that the unique optimal solution[1] to problem (16) is

$$\tilde{C}_n = \left(\tilde{c}_{n,1}, \tilde{c}_{n,2}, \ldots, \tilde{c}_{n,K_n}\right),\ n = 1, \ldots, N \tag{33}$$

The resultant network benefit for the nth VNO is

$$\tilde{\pi}_n = \sum_{k=1}^{K_n} \ln\left(\tilde{c}_{n,k} w_0 \log_2\left(1 + b_{n,k}\frac{p_{n,k}\cdot g_{n,k}}{\tilde{c}_{n,k} w_0 n_0}\right)\right) - \alpha \cdot \sum_{k=1}^{K_n}\tilde{c}_{n,k},\ n = 1, \ldots, N \tag{34}$$

We need to prove that Eq. (33) is also a solution (but not necessarily the unique one) to problem (15). This can be proved by following contradiction.

Assume that Eq. (33) is not a solution to problem (15). According to the Pareto optimality, there must exists another VR allocation vector

$$\hat{C}_n = \left(\hat{c}_{n,1}, \hat{c}_{n,2}, \ldots, \hat{c}_{n,K_n}\right),\ n = 1, \ldots, N \tag{35}$$

different from Eq. (33) which can increases the benefits of part of the VNOs without decreasing the benefits for the other VNOs. Without loss of generality, we assume that, with the new VR allocation vector (35), the benefits of the $N-1$ VNOs remain unchanged except for the $\overline{n}$ th VNO. That means

$$\hat{\pi}_n = \tilde{\pi}_n \text{ for } n = 1, \ldots, N \text{ and } n \neq \overline{n} \tag{36}$$

and

$$\hat{\pi}_{\overline{n}} > \tilde{\pi}_{\overline{n}} \text{ for } n = \overline{n} \tag{37}$$

On condition that Proposition 4 is satisfied, π_n would increase monotonically with $c_{n,k}$, for $\forall\, k \in \{1, \ldots, K_n\}$. Thus, to increase $\tilde{\pi}_{\overline{n}}$, we have to allocate more VR to the $\overline{n}$ th VNO. As the total amount of VR available for the N VNOs is fixed, one or some of the $N-1$ VNOs (other than the $\overline{n}$ th VNO) would receive reduced amount of VR. This contradicts with the assumption that the VR allocation vector (35) is Pareto optimality.

[1] The proof for the uniqueness is given in Proposition 5.

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